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A remark on Stirling's formula

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We shall prove Stirling's formula by showing that for $n = 1, 2, \ldots$

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \cdot e^{r_n} \tag{1}$$

where r_n satisfies the double inequality

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}. \tag{2}$$

The usual textbook proofs replace the first inequality in (2) by the weaker inequality

$$0 < r_n$$

or

$$\frac{1}{12n+6} < r_n.$$

Proof. Let

$$S_n = \log(n!) = \sum_{p=1}^{n-1} \log(p+1)$$

and write

$$\log(p+1) = A_p + b_p - \epsilon_p \tag{3}$$

where

$$A_p = \int_p^{p+1} \log x \, \mathrm{d}x, \quad b_p = \frac{1}{2}[\log(p+1) - \log p]$$
$$\epsilon_p = \int_p^{p+1} \log x \, \mathrm{d}x - \frac{1}{2}[\log(p+1) + \log p].$$

The partition (3) of $\log(p+1)$, regarded as the area of a rectangle with base $(p, p+1)$ and height $\log(p+1)$, into a curvilinear area, a triangle, and a small sliver[1] is suggested by the geometry of the curve $y = \log x$. Then

$$S_n = \sum_{p=1}^{n-1} (A_p + b_p - \epsilon_p) = \int_1^n \log x \, \mathrm{d}x + \frac{1}{2} \log n - \sum_{p=1}^{n-1} \epsilon_p.$$

[1]Taken from G. Darmois, *Statistique Mathématique*, Paris, 1928, pp. 315-317. The only novelty of the present note is the inequality (7) which permits the first part of the estimate (2).

Since $\int \log x \, \mathrm{d}x = x \log x - x$ we can write

$$S_n = \left(n + \frac{1}{2}\right) \log n - n + 1 - \sum_{p=1}^{n-1} \epsilon_p \tag{4}$$

where

$$\epsilon_p = \frac{2p+1}{2} \log\left(\frac{p+1}{p}\right) - 1.$$

Using the well known series

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

valid for $|x| < 1$, and setting $x = (2p+1)^{-1}$, so that $(1+x)/(1-x) = (p+1)/p$, we find that

$$\epsilon_p = \frac{1}{3(2p+1)^2} + \frac{1}{5(2p+1)^4} + \frac{1}{7(2p+1)^6} + \cdots \tag{5}$$

We can therefore bound ϵ_p above and below:

$$\begin{aligned} \epsilon_p &< \frac{1}{3(2p+1)^2}\left\{1 + \frac{1}{(2p+1)^2} + \frac{1}{(2p+1)^4} + \cdots\right\} \\ &= \frac{1}{3(2p+1)^2} \cdot \frac{1}{1 - \frac{1}{(2p+1)^2}} = \frac{1}{12}\left(\frac{1}{p} - \frac{1}{p+1}\right), \end{aligned} \tag{6}$$

$$\begin{aligned} \epsilon_p &> \frac{1}{3(2p+1)^2}\left\{1 + \frac{1}{3(2p+1)^2} + \frac{1}{\left[3(2p+1)^2\right]^2} + \cdots\right\} \\ &= \frac{1}{3(2p+1)^2} \cdot \frac{1}{1 - \frac{1}{3(2p+1)^2}} > \frac{1}{12}\left(\frac{1}{p + \frac{1}{12}} - \frac{1}{p + 1 + \frac{1}{12}}\right). \end{aligned} \tag{7}$$

Now define

$$B = \sum_{p=1}^{\infty} \epsilon_p, \quad r_n = \sum_{p=n}^{\infty} \epsilon_p \tag{8}$$

where from (6) and (7) we have

$$\frac{1}{13} < B < \frac{1}{12}. \tag{9}$$

Then we can write (4) in the form

$$S_n = \left(n + \frac{1}{2}\right) \log n - n + 1 - B + r_n,$$

or, setting $C = e^{1-B}$, as

$$n! = C \cdot n^{n+\frac{1}{2}} e^{-n} \cdot e^{r_n},$$

where r_n is defined by (8), ϵ_p by (5), and from (6) and (7) we have

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

The constant C, known from (9) to lie between $e^{11/12}$ and $e^{12/13}$, may be shown by one of the usual methods to have the value $\sqrt{2\pi}$. This completes the proof. $\square$

The preceding derivation was motivated by the geometrically suggestive partition (3). The editor has pointed out that the inequalities (6) and (7) permit the following brief proof[2] of (2). Let

$$u_n = n! n^{-\left(n+\frac{1}{2}\right)} e^n$$

Then the series

$$\log\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} = 1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \cdots$$

together with (6) and (7) yield the inequalities

$$\exp\left\{\frac{1}{12}\left(\frac{1}{n+\frac{1}{12}} - \frac{1}{n+1+\frac{1}{12}}\right)\right\} < \frac{u_n}{u_{n+1}} = e^{-1}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$$
$$< \exp\left\{\frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right\}$$

Hence

$$v_n = u_n e^{-1/12n}$$

increases and

$$w_n = u_n e^{-1/(12n+1)}$$

decreases, while

$$v_n < w_n = v_n e^{1/12n(12n+1)}.$$

Since

$$v_1 = e^{11/12}, \quad w_1 = e^{12/13}$$

it follows that

$$v_n \to C, \quad w_n \to C, \quad v_n < C < w_n, \quad e^{11/12} < C < e^{12/13}.$$

Thus

$$u_n = Ce^{r_n}$$

where r_n satisfies (2).

[2] A modification of that attributed to Cesàro by A. Fisher, *Mathematical theory of probabilities*, New York, 1936, pp. 93-95.