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Systems, Control

On Linearization of Control Systems

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Summary.

Necessary and sufficient conditions are given for the linearization of control system of the form

$$\dot{x} = f(x) + \sum u_i g_i(x)$$

under a class of transformations including changing of coordinates in the state and input spaces and feedback.

1 Introduction

Consider a class of nonlinear control systems of the form

$$\dot{x} = f(x) + \sum_{i=1}^{k} u_i g_i(x), \tag{1.1}$$

where $x \in R^n$; $f, g_1, \ldots, g_k$ are C^∞ vector fields on R^n and $f(0) = 0$. Let G be a group of transformations of systems (1.1) generated by

(i) change of coordinates in the state space $f, g_i \to \varphi_*(f), \varphi_*(g_i)$, where $\varphi : R^n \to R^n$ is a diffeomorphism preserving $0 \in R^n$.

(ii) linear change of coordinates in the input space R^k, nonlinearly depending on x:

$$g_i \to \sum_{j=1}^{k} h_{ij} g_j, \quad i = 1, \ldots, k, \text{ where } H(x) = (h_{ij}(x))$$

is a $k \times k$ matrix of class C^∞, nonsingular at $0 \in R^n$.

(iii) feedback of the form: $f \to f + \sum_{i=1}^{k} \alpha_i g_i$, where α_i are C^∞ functions, $\alpha_i : R^n \to R$ and $\alpha_i(0) = 0$

We consider a problem of linearizing system (1.1) by means of transformations from the group G, i.e. transforming it via (i)-(iii) in a neighbourhood of zero to the form

$$\dot{x} = Ax + \sum_{i=1}^{k} u_i b_i \tag{1.2}$$

where matrix A and vector fields b_i, $i = 1, \ldots, k$ are constant (in fact we are interested in controllable systems only). We give necessary and sufficient conditions for system (1.1) to be locally linearizable.

Our paper has been inspired by the paper of Brockett [1], where a criterion of linearization of system (1.1) with scalar control is given. Our criterion is similar to that of [1], however Brockett considers a slightly different equivalence relation (the functions $h_{ij}(x)$ in (ii) are assumed to be constant).

The class of transformations G is a natural modification of the class used by Brunovsky [2] for the classification of linear controllable systems (1.2). Our linearization procedure leads to the Brunovsky canonical form and, as in the linear case the transformations can be taken linear, it gives also a proof of his theorem.

For the problem of linearization of systems (1.1) under the transformations (i) the reader is referred to [4][1] and [5] (in the latter paper the case $f = 0$ is considered). The problem of generic classification of systems (1.1) with $f = 0$ under the transformations (i) and (ii) was studied in [3].

2 The main result

Let $V(R^n)$ denote the family of all C^∞ vector fields on R^n. For $f, g \in V(R^n)$ denote $\operatorname{ad}_f g = [f, g]$ and inductively $\operatorname{ad}_f^i g = [f, \operatorname{ad}_f^{i-1} g]$, $i = 1, 2, \ldots$, where $[\cdot,\cdot]$ denotes the Lie bracket. To state the theorem we shall need the following conditions. Below j denotes a nonnegative integer satisfying $0 \leqslant j \leqslant n-1$.

(A1) For any $0 \leqslant p \leqslant j$ and $1 \leqslant s, t \leqslant k$ there are functions $a_{iq} \in C^\infty(R^n)$ such that $\left[\operatorname{ad}_f^p g_s, \operatorname{ad}_f^j g_t\right] = \sum_{\substack{0 \leqslant q \leqslant j \\ 1 \leqslant i \leqslant k}} \alpha_{iq} \operatorname{ad}_f^q g_i$

$$\dim \operatorname{span}\left\{\operatorname{ad}_f^q g_i(x) \mid 0 \leqslant q \leqslant j, 1 \leqslant i \leqslant k\right\} = r_j(x) = \text{const.} \tag{A2}$$

$$\dim \operatorname{span}\left\{\operatorname{ad}_f^q g_i(x) \mid 0 \leqslant q \leqslant n-1, 1 \leqslant i \leqslant k\right\} = r_{n-1}(x) = n. \tag{A3}$$

Those conditions we express now in an invariant form. For any subset $A \subset V(R^n)$ we denote by $\langle A \rangle$ a submodule of $V(R^n)$ generated by A over the ring of smooth functions. Put $\mathscr{G} = \langle g_1, \ldots, g_k \rangle$, $\mathscr{G}_f = f + \mathscr{G} = \{f + g \mid g \in \mathscr{G}\}$ and by induction

$$\mathscr{M}^0 = \mathscr{G}, \quad \mathscr{M}^j = \left\langle \left[\mathscr{G}_f, \mathscr{M}^{j-1}\right], \mathscr{M}^{j-1} \right\rangle,$$

[1] In [4] the linearization condition is stated with an error.

where $[A,B]=\{[a,b]\mid a\in A, b\in B\}$ for any $A,B\subset V(R^n)$. Let $\mathscr{M}^j(x)=\{g(x)\mid g\in\mathscr{M}^j\}\subset T_xR^n$ and denote $m_j(x)=\dim\mathscr{M}^j(x)$. It is easy to see that subsets $\mathscr{G}$, $\mathscr{G}_f\subset V(R^n)$ are invariant under the transformations (ii), (iii). Thus, since the Lie bracket is invariant under (i), the following conditions are invariant under the group G (j denotes a nonnegative integer satysfying $0\leqslant j\leqslant n-1$)

$$\text{(B1)}\qquad \begin{gathered}\mathscr{M}^j \text{ are closed with respect to Lie bracket i.e.}\\ g',g''\in\mathscr{M}^j\Rightarrow[g',g'']\in\mathscr{M}^j,\end{gathered}$$

$$\text{(B2)}\qquad m_j(x)=\text{const},$$

$$\text{(B3)}\qquad m_{n-1}(x)=n.$$

We will use the following notations $p_0(x)=r_0(x)$ and $p_j(x)=r_j(x)-r_{j-1}(x)$ for $j=1,\dots,n-1$. It can be easily seen that $p_0(x)\geqslant p_j(x)\geqslant\ldots\geqslant p_{n-1}(x)\geqslant 0$.

To introduce the Brunovsky canonical form let us denote by N the smallest number such that $r_N(x)=n$. It exists if (A3) holds. It will be proved that condition (B1) implies $r_j(x)=m_j(x)$ and in this case $\{p_j(x)\}$ and the number N can also be defined using $\{m_j(x)\}$ instead of $\{r_j(x)\}$. It is obvious that for $j=N,N+1,\dots,n-1$ we have $r_j(x)=n$.

Let $r_j(x)$ and $p_j(x)$ be constant (it is the case when (A2) is satisfied). We will omit in notations the dependence on x and we will rite r_j and p_j. Let $(x_1,x_2,\dots,x_n)=x$ denote the coordinates of R^n. We shall denote $x=\left(x^0,x^1,\dots,x^N\right)$ where x^0 consists of the first p_0 coordinates of x, x^1 consists of the next p_1 coordinates, ..., x^N consists of the last p_N coordinates of x. In any group x^j denote by $\tilde{x}^j$ the first p_{j+1} coordinates. System (1.1) of the form

$$\text{(2.1)}\qquad \begin{aligned}&g_1=(1,0,\dots,0),\quad g_2=(0,1,0,\dots,0),\quad g_{r_0}=(\underbrace{0,\dots,0}_{(r_0-1)\text{ times}},1,0,\dots,0)\\ &g_i=(0,0,\dots,0)\quad i=r_0+1,\dots,k,\\ &f=(\underbrace{0,\dots,0}_{r_0\text{ times}},\tilde{x}^0,\tilde{x}^1,\dots,\tilde{x}^{N-1})\end{aligned}$$

will be called system in the Brunovsky canonical form.

Theorem 1. *The following conditions are equivalent, locally around $0\in R^n$:*

(a) system (1.1) *is G-linearizable to a controllable system* (1.2)*,*

(b) system (1.1) *is G-equivalent to a system in the Brunovsky canonical form,*

(c) conditions (A1)-(A3) *are satisfied,*

(d) conditions (B1)-(B3) *are satisfied.*

The Theorem can also be formulated in a more abstract way. Namely

Corollary 1. *Let $\mathscr{G}$ be a finitely generated submodule of $V(R^n)$ and $\mathscr{G}_f = f + \mathscr{G}$, where $f \in V(R^n)$, $f(0) = 0$. Then there exists a system of coordinates in a neighbourhood of 0 such that $\mathscr{G}$ and $\mathscr{G}_f$ can be expresed by $\langle g_1', \ldots, g_k' \rangle$ and $f' + \mathscr{G}'$, respectively with f' and g_i' of the form* (2.1) *if and only if the conditions* (B1)-(B3) *are satisfied in a neighbourhood of 0.*

Remark 1. *Conditions* (A1), (A3) *are straightforward modifications of the conditions of Brockett [1]. In [1] the sum on the right hand side of* (A1) *is taken only for $g < j$. This is due to the fact that Brockett assumed the functions $h_{ij}(x)$ in* (ii) *to be constant. In the scalar control case (like in* [1]*) the condition* (A2) *follows from* (A3).

Remark 2. *The Theorem also holds for the class of systems* (1.1) *not satisfying $f(0) = 0$. In this case we have to use a larger class of transformations admitting in* (iii) *α_i not equal zero at zero. The field f in* (2.1) *is then, in general, a sum of constant vector field and that of* (2.1).

In the proof of Theorem 1 we shall use the following modification of the Frobenius theorem.

Lemma 1. *Let $D_0 \subset D_1 \subset \ldots \subset D_N$ denote a sequence of involutive C^∞ distributions on a manifold M ($\dim M = n$) having constant dimensions $\mu_0 \leqslant \mu_1 \leqslant \ldots \leqslant \mu_N$, respectively. Then, around any point $x_0 \in M$, there exists a coordinate system $(x_1, \ldots, x_n)$ such that the integral manifolds of D_i around x_0 are of the form*

$$x_i = C_i, \quad i = \mu_j + 1, \ldots, n. \quad C_i \textit{ constant.}$$

Proof. Let $f_1, \ldots, f_{\mu_0}$ be vector fields which generate the distribution D_0 around x_0, $f_1, \ldots, f_{\mu_1}$ generate D_1, ..., $f_1, \ldots, f_{\mu_N}$ generate the distribution D_N around x_0. Choose vector fields $f_{\mu_N+1}, \ldots, f_n$ in such a way that $f_1, \ldots, f_n$ generate $T_{x_0}M$.

Consider the map T

$$(t_1, \ldots, t_n) \xrightarrow{T} e^{t_1 f_1} \circ \ldots \circ e^{t_n f_n}(x_0) \in M,$$

where $e^{t_i f_i}$ denotes the (local) flow generated by f_i. Let V denote a cube neighbourhood of $0 \in R^n$ such that $T : V \to U = T(V)$ is a diffeomorphism. Then $\left(U, T^{-1}\right)$ is a desired coordinate system.

According to the Frobenius theorem D_j defines in U a foliation $\mathscr{A}$ with μ_j dimensional leaves (maximal integral manifolds of D_j). These leaves are exactly the leaves of the foliation $\mathscr{B}$ constructed as follows. For any point $y \in U$ of the form $y = e^{t_{\mu_{j+1}} f_{\mu_{j+1}}} \circ \ldots \circ e^{t_n fn}(x_0)$ we define a submanifold N_Y which passes through y by

$$N = \left\{ p \in U \mid p = e^{t_1 f_1} \circ \ldots \circ e^{t_{\mu_j} f_{\mu_j}}(y) \right\}$$

(it is a μ_j dimensional submanifold by the regularity of T).

To prove the coincidence of $\mathscr{A}$ and $\mathscr{B}$ note that integral manifolds of D_j are equal to orbits of the family of vector fields generating D_j (Sussmann [6]), so N_Y is contained in the maximal integral manifold of D_j passing through y. Moreover, N_{Y}

is closed (trivial) and open (by the equality of dimensions) subset of the maximal integral manifold of D_j passing through y. Therefore, they are equal.

Since T is bijective it follows that $Z_1, Z_2 \in U$ belong to the same leaf of the foliation $\mathscr{B}$ if and only if $t_i(Z_1) = t_i(Z_2)$ $i = \mu_j + 1, \ldots, n$. This and the fact that $\mathscr{A}$ and $\mathscr{B}$ coincide concludes the proof that $t_1, \ldots, t_n$ are the desired coordinates. $\square$

3 Proof of Theorem 1

The implication (b) $\Rightarrow$ (a) is trivial. To prove (a) $\Rightarrow$ (d) let us notice that condition (d) is G-invariant and for the linear system (1.2) we have $\mathscr{G} = \langle b_1, \ldots, b_k \rangle$, $\mathscr{M}^j = \langle A^q b_i \mid 0 \leqslant q \leqslant j, 1 \leqslant i \leqslant k \rangle$, i.e. for this system condition (d) is satisfied.

(d) $\Rightarrow$ (c). We shall prove using an induction argument with respect to j that (B1), $j = 0, \ldots, n-1$ imply the equalities

$$\mathscr{M}^j = \left\langle \operatorname{ad}_f^q g_i \mid 0 \leqslant q \leqslant j, \quad 1 \leqslant i \leqslant k \right\rangle, \quad j = 0, \ldots, n-1 \tag{3.1}$$

For $j = 0$ this is trivial. Assume that this is true for $j-1$. The inclusion "$\supset$" follows from the definition of $\mathscr{M}^j$. To prove the converse one, note that from (B1) we have $\left[\mathscr{G}, \mathscr{M}^{j-1}\right] \subset \mathscr{M}^{j-1}$. Thus $\left[\mathscr{G}_f, \mathscr{M}^{j-1}\right] = \left[f + \mathscr{G}, \mathscr{M}^{j-1}\right] \subset \left[f, \mathscr{M}^{j-1}\right] + \left[\mathscr{G}, \mathscr{M}^{j-1}\right] \subset \left[f, \mathscr{M}^{j-1}\right] + \mathscr{M}^{j-1}$. Therefore

$$\mathscr{M}^j = \left\langle \left[\mathscr{G}_f, \mathscr{M}^{j-1}\right], \mathscr{M}^{j-1} \right\rangle \subset \left\langle \left[f, \mathscr{M}^{j-1}\right], \mathscr{M}^{j-1} \right\rangle.$$

This and the induction assumption on the form of $\mathscr{M}^{j-1}$ give the desired inclusion.

The conditions (B1)-(B3) and equality (3.1) imply easily the conditions (A1)-(A3).

(c) $\Rightarrow$ (b). We denote $f = \left(f^0, f^1, \ldots, f^N\right)$, where f^0 consists of the first p_0 coordinates of $f = (f_1, \ldots, f_n)$, f^1 consists of the next p_1 coordinates, ..., f^N consists of the last p_N coordinates. We shall also use the notations from (2.1).

The distributions D_j generated by $R_j = \left\{ \operatorname{ad}_f^q g_i \mid 0 \leqslant q \leqslant j, 1 \leqslant i \leqslant k \right\}$ satisfy the assumptions of Lemma 1. Let us choose the coordinates around $0 \in R^n$ in such a way that the integral manifolds of D_j, $j = 0, \ldots, N$ are given by $x_i = c_i$, $i = r_j + 1, \ldots, n$. We assume that f and g_i's in (1.1) are expressed in this coordinate system. All our future changes of coordinates will preserve the above integral manifolds. The form of the integral manifolds of D_j implies that the vector fields in R_j have zero components along the last $n - r_j$ coordinates. Using this fact and $\dim \operatorname{span} R_j(x) = r_j$ we shall prove that the coordinates of f^j do not depend on the variables $x^0, \ldots, x^{j-2}$ $(j \geqslant 2)$ and $\operatorname{rank} \frac{\partial f^j}{\partial x^{j-1}}(x) = p_j$. Note first that for $g \in R_{j-1}$ we have that g and so $Dg \cdot f$ are equal to zero along the last $n - r_{j-1}$ coordinates. Thus, along these coordinates $[f, g]$ is equal to $-Df \cdot g$. In particular if $g' \in R_{j-2}$ then we have $Df^j g = 0$ by the fact that $[f, g'] \in R_{j-1}$. This gives that f^j does not depend on $x^0, \ldots, x^{j-2}$. The equality $\frac{\partial f^j}{\partial x^{j-1}}(x) = p_j$ is a consequence of the fact that there are p_j independent vectors among $[f, g]$, $g \in R_{j-1}$.

Now we shall transform the vector field $f = \left(f^0, \ldots, f^N\right)$ to the canonical form (2.1). We shall use the same notations to express it in an old coordinate system and in a new one. In the first step we introduce the new coordinates $y^{N-1} = \big(f^N, \tilde{\tilde{x}}^{N-1}\big)$, $y^j = x^j$ if $j \neq N-1$, where $\tilde{\tilde{x}}^{N-1}$ denotes $p_{N-1} - p_N$ coordinates of x^{N-1} chosen in such a way that rank $\frac{\partial y^{N-1}}{\partial x^{N-1}}(0) = p_{N-1}$ (this is possible since rank $\frac{\partial f^N}{\partial x^{N-1}}(0) = p_N$). We obtain f in the form $\big(f^0, \ldots, f^{N-1}, \tilde{x}^{N-1}\big)$, where we write $\tilde{x}^{N-1}$ instead of $\tilde{y}^{N-1}$ to use consequently notation x for coordinates. In the second step we introduce the new coordinates $y^{N-2} = \big(f^{N-1}, \tilde{\tilde{y}}^{N-2}\big)$ and get $f = \big(f^0, \ldots, f^{N-2}, \tilde{x}^{N-2}, \tilde{x}^{N-1}\big)$. After $N-1$ steps we obtain $f = \big(f^0, \tilde{x}^0, \ldots, \tilde{x}^{N-1}\big)$.

By the form of the distribution D_0 we can transform the vector fields g_i, $i = 1, \ldots, k$ to the canonical form $g_i = (\underbrace{0, \ldots, 0}_{(i-1)\ \text{times}}, 1, 0, \ldots, 0)$ for $f = \big(0, \tilde{x}^0, \ldots, \tilde{x}^{N-1}\big)$. The proof is complete.

Added in proof. After this paper had been submitted, R. W. Brockett informed us that he had obtained an analogous result.

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